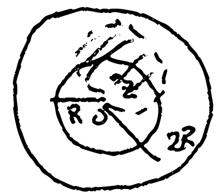
Lecture 18 (11/5/21).

Def. (1) An analytic fcn f in $\mathbb{C}$ is called <u>entire</u>.

- By previous results, f is entire $\Leftrightarrow$ $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$, w/ R.O.C. $= \infty$.

- Moreover, by Cauchy's Estimate, if $M_r := \sup_{|z| \le r} |f(z)|$, then for $|z| \le R$

$$|f'(z)| \le \frac{n!\, M_{2R}}{R} \quad (1)$$



Thus, if $|f| \le M$ for <u>all</u> $z \in \mathbb{C}$, then $M_r \le M, \forall r$, and (1) $\Rightarrow$ $f'(z) = 0$ for <u>all</u> $z \in \mathbb{C}$ $\Rightarrow$ f is constant.

<u>Liouville's Thm</u> If f is entire and bounded, then f is constant.

Fundamental Thm of Algebra.

Every nonconstant polynomial $p(z) = \sum_{n=0}^{N} p_n z^n$ has at least one root.

<u>Pf.</u> Suppose not. Then $f(z) = 1/p(z)$ is entire and $\lim_{|z| \to \infty} |f(z)| = 0$, since

$p(z) = z^N(p_N + O(\frac{1}{z}))$ and WLOG $p_N \neq 0$.

By Liouville, f is constant $\Rightarrow$ p is constant. $\blacksquare$

Thus, by Euclid, $p(z) = (z-a_1)q(z)$, where $\deg q = N-1$. By induction, $\exists\, a_1, \dots, a_N$ and $c \neq 0$ s.t.

$$p(z) = c(z-a_1)\dots(z-a_N)$$

$$= c\,(z-r_1)^{m_1}\dots(z-r_d)^{m_d}$$

where $r_j \in \{a_1, \dots, a_N\}$ are distinct and the multiplicities $m_1 + \dots + m_d = N$.

Rem. Although entire fcns can be regarded as "polynomials of ∞ degree", they can behave quite differently from polynomials.

Ex. ① $f(z)=e^z$ is entire, non-constant, but $f(z)\neq 0$, $\forall z\in\mathbb{C}$. Moreover, if $z=x\in\mathbb{R}$ $f(x)=e^x\to 0$ as $x\to-\infty$ and $f(x)\to\infty$ as $x\to+\infty$. Thus, $\lim_{z\to\infty}|f(z)|$ does not exist.

Def. ② If f is analytic in G and $z_0\in G$ is s.t. $f(z_0)=0$, then z_0 is a zero. If $\exists\, m\in\mathbb{Z}_+$ s.t. $f(z)=(z-z_0)^m g(z)$ in $B(z_0,r)\subseteq G$, where g is analytic and $g\neq 0$ in $B(z_0,r)$, then the zero z_0 has multiplicity m.

<u>Prop1.</u> If f is analytic and $f(z_0)=0$, then z_0 has multiplicity $m \Leftrightarrow f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$.

<u>Pf.</u> DIY. $\boxtimes$

<u>Thm1.</u> Let f be analytic in connected $G \subseteq \mathbb{C}$. TFAE:

(i) $f \equiv 0$.

(ii) $\exists z_0$ s.t. $f^{(n)}(z_0) = 0, \forall n \in \mathbb{Z}_+$.

(iii) $Z_f = \{z : f(z) = 0\}$ has a limit point in G.

<u>Pf.</u> Clearly, (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii).
We show (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii). This completes the pf.

(ii) $\Rightarrow$ (i). By previous $\exists B(z_0, r) \subseteq G$ and
$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$ in $B(z_0, r)$. Thus, assumption $\Rightarrow f \equiv 0$ in $B(z_0, r)$. Let
$A = \{z \in G : f^{(n)}(z) = 0, \forall n\}$. We have just shown that $A \neq \emptyset$ and A is open.

A is also closed by continuity of $f^{(n)}$
$\Rightarrow A = G$ by connectedness of G. This
proves (ii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (ii). Let z_0 be limit pt of Z_f and
$B(z_0, r) \subseteq G$. Suppose $f^{(n)}(z_0) \neq 0$ for some
n. Let m be smallest such number.
($f(z_0) = 0$ by cont. $\Rightarrow m$ is well defined.)
Then, $f(z_0) = \dots = f^{(m-1)}(z_0) = 0$ and
$f^{(m)}(z_0) \neq 0$. By p.s. expansion,

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n \\
&= \sum_{n=m}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n \\
&= (z-z_0)^m \underbrace{\sum_{k=0}^{\infty} a_k (z-z_0)^k}_{\text{same R.O.C.}}
\end{aligned}
$$

and $a_0 = \frac{1}{m!} f^{(m)}(z_0) \neq 0$.

Thus, $f(z) = (z-z_0)^m g(z)$ w/ $g \neq 0$ in $B(a,r)$
(by shrinking r if necessary.)

In particular, the zero z_0 has multip. m and $f(z) \neq 0$ in $B(a,r)$. This contradicts z_0 being limit pt of Z_f, and hence $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{Z}_+$, i.e. (ii) holds.

This completes the pf of Thm 1. ☒

<u>Cor 1</u>. Let $f \not\equiv 0$ be analytic in connected $G \subseteq \mathbb{C}$. Every zero $z=a$ of f is <u>isolated</u> ($\exists B(a,r)$ s.t. $f(z) \neq 0$ in $B(a,r) \setminus \{a\}$) and has finite multiplicity ($\exists m$ s.t. $f(z) = (z-a)^m g(z)$, $g(a) \neq 0$).

Max Modulus Thm. Let f be analytic in connected $G \subseteq \mathbb{C}$. If $\exists\, a \in G$ s.t.
$|f(a)| = \sup_{z \in G} |f(z)|$ (exists a max for $|f|$),
then f is constant.

Pf. Suppose such $a \in G$ exists, and let $B(a,R) \subseteq G$. By CIF, $\forall\, 0 < r < R$

$$f(a) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z-a}\,dz, \quad \gamma_r(s) = a + re^{is},$$

$$\Rightarrow f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{is})\,ds \Rightarrow$$

$$|f(a)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(a+re^{is})|\,ds, \text{ i.e.}$$

$|f(a)| \le$ average of $|f|$ on γ_r $\Rightarrow$ $\{|f(a)|$ is max!$\}$ $\Rightarrow$ $|f|$ is constant on γ_r.

$0<r<R$ arbitrary $\Rightarrow$ $|f|$ is constant on $B(a,R)$.

<u>Prop. 1</u> If f is analytic on $B(a,R)$ and maps $B(a,R)$ into circle/line, then f is constant.

Pf. By composing w/ Möbius, WLOG $f(B(a,R)) \subseteq \mathbb{R}$.

$f = u+iv \Rightarrow v \equiv 0$ on $B(a,R) \Rightarrow$ {CR eq.}

$u \equiv$ constant $\Rightarrow u+iv$ is const. ⊠

Thus, by Prop 1, f is constant on $B(a,R)$. If α is that constant, $f-\alpha \equiv 0$ on $B(a,R)$ so by Thm 1 (iii) or Cor 1, $f \equiv \alpha$.

This completes pf of MMT. ⊠